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Analytical study of steady-state compressible flow of perfect gas with constant heat flux and friction in constant area ducts

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A B S T R A C T

Exact analytical solutions of one-dimensional gas dynamics are intensively applied in engineering practice as a tool in modelling and simulating the piping systems that utilize a compressible medium as their working fluids. Well-known exact analytical solutions for simple types of flows, i.e. for flow processes in which only a single effect is taken into account (e.g. such limiting cases of flows as isentropic, adiabatic or isothermal), are classics of modern one-dimensional gas dynamics theory formed in the first half of last century. At present, gas dynamics does not possess an exact analytical solution for more than a single factor bringing about changes in fluid properties. In this paper the possibility of obtaining general and particular solutions of a non-linear ordinary differential equation (ODE) system describing one-dimensional steady-state flow of compressible ideal gas in constant area ducts with a constant heat flux and friction factor is discussed. It shows that ODE system variables can be separated, and integrals can be taken in terms of elementary functions. Since an analytical solution is the most important result of the paper, its detailed derivation is presented. The mathematical properties of the solution are analyzed in order to calculate this type of compressible flow. For the purpose of this analysis, the functions of the solution and duct performance characteristics of the flow model are demonstrated for various flows and heat flux values, as well as distributions of flow parameters along the duct for both subsonic and supersonic flows. The thermodynamic constraints of the solution are also studied. The analytical solution formulas obtained in this paper may serve as a definition of heat flux and friction factor in ducts from a viewpoint of one-dimensional gas dynamics.

1. Introduction

Every thermal system (TS) consists of different types of elements among which are heat exchangers, compressors, turbines, ducts, etc. They are applied in heat exchanger networks, aircraft air-conditioning systems, gas transport, cryogenic systems, and others. The flow in such systems as a rule is unsteady, yet sometimes however for the purpose of modelling a steady-state model is more appropriate, e.g. for solving problems associated with simulation of not large systems with high speed flows, for TS optimal design or/and control. The steady-state models are also used to obtain initial conditions for transient flow simulation. Because a duct occupies a particular place in engineering, its model often extends to modelling other types of TS elements such as bends, elbows, diffusers, confusers, heat exchanger channels, and the like. Such an extension to the model can be achieved by introducing correction factors to the one-dimensional model of a constant cross-sectional area duct. One-dimensional models of duct flow are widely reported in literature devoted to thermodynamics, as well as to gas dynamics. Engineering practice provides evidence of the adequacy of one-dimensional approach to modelling TS. The objective of this paper is to study the one-dimensional steady-state model of flow process in ducts. To proceed, we first formulate basic equations of one-dimensional steady-state compressible flow of a perfect gas in a constant cross-sectional area duct. There are five such equations [1–5]:

continuity equation

$$\frac{d(\rho v A_c)}{dx} = 0, \tag{1}$$

momentum equation

$$v dv + \frac{dP}{\rho} = f_x dx - \frac{4\tau_w}{\rho D_h} dx, \tag{2}$$

energy equation

$$\frac{d}{dx}\left(c_p T + \frac{1}{2} v^2\right) = \frac{q_l}{\dot{m}} + f_x = q_x + f_x = w_x. \tag{3}$$

perfect gas equation of state

$$P = \rho R T. \tag{4}$$

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Nomenclature

a	sound speed, m/s
A_c	cross-sectional area of a duct, m^2
c_p	heat capacity, J/kg K
D_h	duct diameter, m
f_x	mass force per mass unit, N/kg
g	gravitational constant, m/s^2
L	length, m
M	Mach number
$\dot{m}$	mass flow rate, kg/s
P	pressure, Pa
q_l	heat flux, W/m
q_x	heat flux per mass flow rate unit, W/(m kg/s)
R	ideal gas constant, J/kg K
T	temperature, K
v	gas velocity, m/s
x	spatial coordinate, m

Greek symbols

ρ	density, kg/m^3
λ	Darcy–Weisbach friction factor
τ_w	shear stress, N/m^2

The formula for τ_w may be used in the form of the Fanning or Darcy–Weisbach friction. In the latter case we have

$$\tau_w = \frac{1}{8}\lambda\rho v^2. \quad (5)$$

The equations Eqs. (1)–(5) and their particular solutions constitute an essential part of one-dimensional gas dynamics theory [2–5]. The particular solutions are commonly used in engineering practice. Several limiting cases of flows from equations Eqs. (1)–(5) have been thoroughly studied. Provided that a contribution of kinetic energy is small enough, Eqs. (2)–(3) can be solved independently. As a result we obtain a solution that is widely used in modelling TS, provided that Mach number $M < 0.3$. In this case, gas is considered to be an incompressible fluid [2–5]. If the contribution of kinetic energy to the flow cannot be neglected, then there exists a group of limiting cases of flows collectively known as simple types of flows [2]. These flows take into account a single effect or none of the effects, e.g. an isentropic flow, isothermal and adiabatic flows with friction, as well as frictionless flow with heat exchange. The models of simple types of flows are non-linear ODE following from Eqs. (1)–(5). Their exact analytical solutions are possible because the variables of these models can be easily separated and integrals taken analytically. The isentropic flow model results from Eqs. (1)–(5) under the assumption that $q_l = f_x = \lambda = 0$. Integrating this model, we can obtain an explicit equation for the flow rate in an idealized nozzle or the Prandtl equation relating gas parameters ahead and after a normal shock [2–5]. The next limiting case of a fluid flow corresponds to the following conditions $f_x = \lambda = 0$. It is valid when the friction heat is small in comparison to the external heat transfer. This model can also be used for studying such types of discontinuities as moisture condensation shock, explosion waves in combustible mixtures, etc. From this model the well-known Chapman–Jouguet rule is developed [2,3] as well. The adiabatic flow formula is obtained provided that a duct is insulated, $q_l = 0$, friction factor is constant, and mass forces are zero, $f_x = 0$. This flow model is valid for reasonably short ducts. If we assume that heat transfer exists, but the flow temperature is constant along the duct, $T = const$, then we deal with isothermal flow. It is often used for the simulation of gas transport over a long distance as well as the simulation of low-pressure gas distribution networks. Working formulas and detailed study of the flow models are presented in [2–5]. However, one more analytical but approximate solution can be found. It can be obtained on condition that the velocity and density in Eq. (2) change linearly along the channel, whereas kinetic energy in Eq. (3) is neglected. In this manner the influence of temperature on channel hydraulics is taken into account, and integrals of Eq. (3) can be taken with various relations for q_l being specific to heat transfer [6–8]. Such a solution is widely used in heat exchanger network analysis and design (see e.g. [9,10]). When integration of Eqs. (1)–(5) is performed on the assumption of constant coefficients, the solution may require a specific technique for its use. Usually this technique is reduced to the channel segmentation, with its own coefficients for each of the segments computed. It allows taking into account the variability of the coefficients along the channel (see e.g. [11]). Engineers often apply such methodology instead of using numerical methods, although, generally, models of the complex flow systems are collections of both numerical and analytical models. The exact analytical solutions of gas dynamics discussed above are a source of variety of semi-empirical models. The necessity of such models stems from the fact that the one-dimensional theory is not valid for modelling of flows in ducts having complex geometries or in constant area ducts in the case where the models of simple types of flows are inaccurate approximations of a real flow process. For example, a discharge coefficient is introduced into the isentropic flow rate formula giving the possibility of simulating such elements as nozzles, orifices, etc [2]. In the case of a sudden enlargement, face pressure ratio is an empirical parameter [12] of the isentropic model. In paper [11] a kinetic energy correction factor is introduced into the exact analytical solution formula of an adiabatic flow with friction. This correction factor is formulated as a ratio of the flow obeying the 1/n-th-power law and one-dimensional flow. The choked flow is a subject of special concern in the analysis and design of flow systems. Since the computation of sonic flow rate is time-consuming, a shortcut formula [13] is proposed for the prediction of mass flow rates under choked conditions through long pipes for both adiabatic and isothermal flows. From the above consideration it follows that the main emphasis has been placed on grounding the adequacy of the use of the chosen form of the analytical solution (e.g. adiabatic or isothermal) to the given conditions of channel operation, and any exact analytical solution that combines gas dynamic mechanisms described by limiting cases of flows has not been reported so far. In this paper the possibility of obtaining general and particular solutions of non-linear ordinary differential equation (ODE) system Eqs. (1)–(5) is demonstrated. It is shown that ODE system variables can be separated and integrals can be taken in terms of elementary functions. The analytical solution obtained considers both friction and heat transfer, with kinetic energy having a considerable influence on the flow process, and can be directly applied to solving problems at these boundary conditions, or can serve as a basis for modelling. For example, the boundary conditions of model Eqs. (1)–(5) are found in electric resistance heating, nuclear heating, and counter-flow heat exchangers, in which two fluids have nearly the same fluid capacity rates and the wall is highly conductive [8], etc. The solution obtained in this paper extends the set of exact analytical solutions of gas dynamics. Because the analytical solution as such is the most important result of this paper, its detailed derivation is provided below.

2. Analytical solution

2.1. General solution

Since a constant area channel is considered, then Eq. (1) can be written as

$$\dot{m} = \rho v A_c. \tag{6}$$

Velocity and density may be expressed in terms of pressure, flow rate and temperature with Eqs. (4), (6) as follows

$$\rho = \frac{P}{RT},\ v = \frac{\dot{m}}{\rho A_c} = \frac{\dot{m}RT}{PA_c},\ \frac{dv}{dx} = \frac{\dot{m}R}{A_c}\left(\frac{1}{P}\frac{dT}{dx} - \frac{T}{P^2}\frac{dP}{dx}\right), \tag{7}$$

We substitute Eq. (7) into the energy equation Eq. (3) to get

$$c_P\frac{dT}{dx} + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T}{P}\left(\frac{1}{P}\frac{dT}{dx} - \frac{T}{P^2}\frac{dP}{dx}\right) = w_x,$$

then collecting the similar terms we have

$$\left[c_P + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T}{P^2}\right]\frac{dT}{dx} = w_x + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T^2}{P^3}\frac{dP}{dx} \tag{8}$$

Now we transform the impulse equation Eq. (2) substituting Eq. (7) and Eq. (5) for τ_w into Eq. (2) to obtain

$$\left(\frac{\dot{m}}{A_c}\right)^2 R\left(\frac{1}{P}\frac{dT}{dx} - \frac{T}{P^2}\frac{dP}{dx}\right) + \frac{dP}{dx} = -\frac{\lambda}{2D_h}\left(\frac{\dot{m}}{A_c}\right)^2\frac{RT}{P} + \frac{P}{RT}f_x.$$

Collecting the similar terms, we have

$$\left(\frac{A_c^2}{\dot{m}^2R} - \frac{T}{P^2}\right)\frac{dP}{dx} = -\frac{1}{P}\frac{dT}{dx} - \frac{\lambda}{2D_h}\frac{T}{P} + \left(\frac{A_c}{\dot{m}R}\right)^2\frac{P}{T}f_x. \tag{9}$$

Expressing a temperature gradient from Eq. (8)

$$\frac{dT}{dx} = \frac{w_x + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T^2}{P^3}\frac{dP}{dx}}{\left[c_P + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T}{P^2}\right]} \tag{10}$$

and substituting it into Eq. (9) gives

$$\left(\frac{A_c^2}{\dot{m}^2R} - \frac{T}{P^2}\right)\frac{dP}{dx} = -\frac{1}{P}\frac{w_x + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T^2}{P^3}\frac{dP}{dx}}{\left[c_p + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T}{P^2}\right]} - \frac{\lambda}{2D_h}\frac{T}{P} + \left(\frac{A_c}{\dot{m}R}\right)^2\frac{P}{T}f_x$$

With algebra the last equation is reduced to

$$\frac{dP}{dx} = \frac{-\frac{1}{P}w_x - \frac{\lambda}{2D_h}\frac{T}{P}\left[c_p + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T}{P^2}\right] + \left(\frac{A_c}{\dot{m}R}\right)^2 f_x\frac{P}{T}\left[c_p + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T}{P^2}\right]}{\left[c_p\frac{A_c^2}{\dot{m}^2R} + (gR - c_p)\frac{T}{P^2}\right]}. \tag{11}$$

Substituting Eq. (11) into Eq. (10) and making some reductions, we obtain

$$\frac{dT}{dx} = \frac{\left[\frac{A_c^2}{\dot{m}^2R} - \frac{T}{P^2}\right]w_x - \frac{\lambda}{2D_h}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{T^3}{P^4} + f_x\frac{T}{P^2}}{\left[c_p\frac{A_c^2}{\dot{m}^2R} + (R - c_p)\frac{T}{P^2}\right]}. \tag{12}$$

Dividing equation Eq. (12) by Eq. (11), we have

$$\frac{dT}{dP} = -\frac{\left[\frac{A_c^2}{\dot{m}^2R} - \frac{T}{P^2}\right]w_x - \frac{\lambda}{2D_h}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{T^3}{P^4} + f_x\frac{T}{P^2}}{\frac{1}{P}w_x + \frac{\lambda}{2D_h}\frac{T}{P}\left[c_p + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T}{P^2}\right] - \left(\frac{A_c}{\dot{m}R}\right)^2 f_x\frac{P}{T}\left[c_p + \left(\frac{\dot{m}R}{A_c}\right)^2\frac{T}{P^2}\right]}. \tag{13}$$

Equation Eq. (13) relates the pressure and temperature at every flow section. If it was solved it would be possible to eliminate one unknown, i.e. to reduce the problem to a single ODE. Making the following change of variables

$$z = \frac{T}{P};\ T = z\cdot P;\ \frac{dT}{dP} = P\frac{dz}{dP} + z, \tag{14}$$

and considerable algebraic manipulation with Eq. (13), we obtain the following equation

$$\frac{dP}{dz} + R(z)\cdot P = Q(z), \tag{15}$$

where expressions of $R(z)$ and $Q(z)$ are

$$R(z) = -\frac{c_p\frac{\lambda}{2D_h}z - c_p\left(\frac{A_c}{\dot{m}R}\right)^2 f_x\frac{1}{z}}{\left\{\left[c_p\left(\frac{F}{\dot{m}R}\right)^2 f_x - \frac{A_c^2}{\dot{m}^2R}w_x\right] - c_p\frac{\lambda}{2D_h}z^2\right\}}$$

$$Q(z) = \frac{w_x + \frac{\lambda}{2D_h}\left(\frac{\dot{m}R}{A_c}\right)^2 z^2 - f_x}{\left\{\left[c_p\left(\frac{A_c}{\dot{m}R}\right)^2 f_x - \frac{A_c^2}{\dot{m}^2R}w_x\right] - c_p\frac{\lambda}{2D_h}z^2\right\}}.$$

It is known [14] that a linear heterogeneous ODE Eq. (15) has the following solution

$$P = e^{-\int R(z)\cdot dz}\left[\int Q(z)\cdot e^{\int R(z)\cdot dz}dz + Const\right], \tag{16}$$

We demonstrate below, that the general solution for the problem Eqs. (1)–(5) or Eqs. (11)–(12) can be obtained. For this purpose we differentiate Eq. (16) with respect to x to get

$$\frac{dP}{dx} = \left[\int Q(z)\cdot e^{\int R(z)\cdot dz}dz + Const\right]\left(e^{-\int R(z)\cdot dz}\right)'\frac{dz}{dx} + e^{-\int R(z)\cdot dz}\left[\int Q(z)\cdot e^{\int R(z)\cdot dz}dz + Const\right]'\frac{dz}{dx}.$$

The above derivative can be rewritten as follows

$$\frac{dP}{dx} = \left[\int Q(z)\cdot e^{\int R(z)\cdot dz}dz + Const\right] \times \left(\int R(z)\cdot dz\right)\cdot\left(\int\frac{\partial R(z)}{\partial x}dz\right)\cdot e^{-\int R(z)\cdot dz}\frac{dz}{dx} + e^{-\int R(z)\cdot dz} \left[\int\frac{\partial Q(z)}{\partial x}\cdot e^{\int R(z)\cdot dz}dz + \int Q(z)\cdot\left(\int R(z)\cdot dz\right) \times\left(\int\frac{\partial R(z)}{\partial x}e^{-\int R(z)\cdot dz}dz\right)\cdot dz\right]\frac{dz}{dx}. \tag{17}$$

Further, we introduce variable z (see Eq. (14)) into Eq. (11)

$$\frac{dP}{dx} = -\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2 \frac{w_x + \frac{\lambda}{2D_h}z\left[c_pP + \left(\frac{\dot{m}R}{A_c}\right)^2 z\right] - \left(\frac{A_c}{\dot{m}R}\right)^2 f_x\frac{1}{z}\left[c_pP + \left(\frac{\dot{m}R}{A_c}\right)^2 z\right]}{\left[c_pP + \left(\frac{\dot{m}R}{A_c}\right)^2 (1-\frac{c_p}{R})z\right]}. \tag{18}$$

Finally, we equate Eq. (17) to Eq. (18) to get

$$\left\{\left[\int Q(z)\cdot e^{\int R(z)\cdot dz}dz + Const\right]\cdot\left(\int R(z)\cdot dz\right)\cdot\left(\int\frac{\partial R(z)}{\partial x}dz\right)\cdot e^{-\int R(z)\cdot dz} + e^{-\int R(z)\cdot dz}\left[\int\frac{\partial Q(z)}{\partial x}\cdot e^{\int R(z)\cdot dz}dz + \int Q(z)\cdot\left(\int R(z)\cdot dz\right)\cdot\left(\int\frac{\partial R(z)}{\partial x}e^{-\int R(z)\cdot dz}dz\right)\cdot dz\right]\right\}\frac{dz}{dx} = -\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2 \frac{w_x + \left\{c_pe^{\int R(z)dz}\left[\int Q(z)e^{-\int R(z)dz}dz + Const\right] + \left(\frac{\dot{m}R}{A_c}\right)^2 z\right\}\cdot\left(\frac{\lambda}{2D_h}z - f_x\frac{1}{z}\right)}{\left[c_pe^{\int R(z)dz}\left[\int Q(z)e^{-\int R(z)dz}dz + Const\right] + \left(1-\frac{c_P}{R}\right)\cdot\left(\frac{\dot{m}R}{A_c}\right)^2 z\right]}.$$

It is evident that the above ODE variables are separable. Consequently, it proves the existence of a general solution to problem Eqs. (1)–(5).

2.2. Particular solutions

Now we try to find the solutions of two integrals of Eq. (16) in terms of elementary functions. The first one has the following form

$$\int R(z)dz = -\int\frac{c_p\frac{\lambda}{2D_h}z - c_p\left(\frac{A_c}{\dot{m}R}\right)^2 f_x\frac{1}{z}}{\left\{\left[c_p\left(\frac{A_c}{\dot{m}R}\right)^2 f_x - \frac{A_c^2}{\dot{m}^2R}w_x\right] - c_p\frac{\lambda}{2D_h}z^2\right\}}dz.$$

It can be shown that the above integral splits into two integrals. The first integral has the form

$$\int R(z)dz = \ln z^{2B}|z^2+A|^{\frac{1-2B}{2}}, \tag{19}$$

where

$$A = \frac{\left(w_x - \frac{c_p}{R}f_x\right)}{\frac{c_p}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{\lambda}{2D_h}},\quad B = -\frac{1}{2}\frac{\frac{c_p}{R}f_x}{\left(w_x - \frac{c_p}{R}f_x\right)}. \tag{20}$$

Substituting the solution of the integral Eq. (19) into Eq. (16), we obtain the following form of the integral

$$\int Q(z)\cdot e^{\int R(z)dz}dz = -\frac{\left(\frac{\dot{m}R}{A_c}\right)^2}{c_p}\int z^{2(B+1)}(z^2+A)^{-\frac{1+2B}{2}}\times dz - \frac{(w_x - f_x)}{c_p\frac{\lambda}{2D_h}}\int z^{2B}\cdot(z^2+A)^{-\frac{1+2B}{2}}dz. \tag{21}$$

The integrals of Eq. (21) have binomial differentials as their integrands, so the approximate analytical solution is only possible for Eqs. (1)–(5). If the contribution of mass forces is negligible, then the integrals of Eqs. (19)–(21) appear to be

$$\int R(z)dz = \int\frac{\frac{c_P}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{\lambda}{2D_h}\cdot z}{q_x + \frac{c_P}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{\lambda}{2D_h}\cdot z^2}dz,$$

$$\int Q(x)\cdot e^{\int R(x)\cdot dx}dx = \int\left[-\frac{\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2 q_x + \frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^4\frac{\lambda}{2D_h}z^2}{q_x + \frac{c_P}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{\lambda}{2D_h}\cdot z^2}\right]\left(\frac{q_x}{\frac{c_P}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{\lambda}{2D_h}} + z^2\right)^{\frac{1}{2}}dz$$

These integrals can be taken in terms of elementary functions that provides the exact analytical solution of Eqs. (1)–(5)

$$\Phi_1 = \frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}P + \frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\left\{\frac{a}{c}\left(1-\frac{b}{2c}\right)\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right| + \frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\frac{b}{c}\frac{z}{2}\right\} = Const \tag{22}$$

or

$$P = -\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{1}{\sqrt{c}}\left\{\frac{a}{\sqrt{a+c\cdot z^2}}\left(1-\frac{b}{2c}\right)\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right| + \frac{b}{\sqrt{c}}\frac{z}{2}\right\} + Const\frac{\sqrt{c}}{\sqrt{a+c\cdot z^2}}, \tag{23}$$

where

$$a = q_x,\quad b = \left(\frac{\dot{m}R}{A_c}\right)^2\frac{\lambda}{2D_h},\quad c = \frac{c_P}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{\lambda}{2D_h}. \tag{24}$$

Further, we multiply Eq. (23) by z

$$zP + \frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{b}{c}\frac{z^2}{2} = -\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{1}{\sqrt{c}}\frac{a\cdot z}{\sqrt{a+c\cdot z^2}}\left(1-\frac{b}{2c}\right)\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right| + Const\frac{z\sqrt{c}}{\sqrt{a+c\cdot z^2}}. \tag{25}$$

If we eliminate velocity with Eq. (7) from the stagnation temperature formula

$$T^* = T + \frac{v^2}{2c_p}, \tag{26}$$

then the right part of Eq. (25) is equivalent to Eq. (26), and Eq. (25) takes the form

$$T^* = -\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{1}{\sqrt{c}}\left\{\frac{a\cdot z}{\sqrt{a+c\cdot z^2}}\left(1-\frac{b}{2c}\right)\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right|\right\}+Const\frac{z\sqrt{c}}{\sqrt{a+c\cdot z^2}} \tag{27}$$

Differentiation of Eq. (27) with respect to x and substitution of the result in the energy equation Eq. (3) expressed in the following form (obtained by substitution of Eq. (26) into Eq. (3))

$$\frac{dT^*}{dx} = \frac{q_x}{c_p} \tag{28}$$

reduces Eq. (1)–(5) to a single differential equation

$$\frac{dT^*}{dx} = \left[-\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\left(1-\frac{b}{2c}\right)\frac{a}{c}\frac{\ln\left|z+\sqrt{z^2+\frac{a}{c}}\right|}{\sqrt{z^2+\frac{a}{c}}}+\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\right.$$
$$\times\left(1-\frac{b}{2c}\right)\frac{a}{c}\frac{z^2\ln\left|z+\sqrt{z^2+\frac{a}{c}}\right|}{\sqrt{\left(z^2+\frac{a}{c}\right)^3}}-\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2$$
$$\times\left(1-\frac{b}{2c}\right)\frac{a}{c}\frac{z}{\left(z^2+\frac{a}{c}\right)}+Const\frac{1}{\sqrt{z^2+\frac{a}{c}}}$$
$$\left.-Const\frac{z^2}{\sqrt{\left(z^2+\frac{a}{c}\right)^3}}\right]\frac{dz}{dx} = \frac{q_x}{c_p}$$

The above ODE is separable. Its indefinite integrals can be easily taken using Tables of Indefinite Integrals [15] as made when solving Eq. (15). Doing so we finally arrive at the following solution

$$\Phi_2 = -\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\left(1-\frac{b}{2c}\right)\frac{a}{c}\frac{z\ln\left(z+\sqrt{\left|z^2+\frac{a}{c}\right|}\right)}{\sqrt{\left|z^2+\frac{a}{c}\right|}}$$
$$+Const\cdot\frac{z}{\sqrt{\left|z^2+\frac{a}{c}\right|}}-\frac{q_x}{c_p}L-C = 0. \tag{29}$$

Note that Eq. (22) and Eq. (29) can be expressed in terms of specific volume $\upsilon = gRz$ instead of z, which is proportional to the gas constant R.

2.2.1. Specific solution at sonic speed

Eqs. (11)–(12) have the same denominator. Let us find out which solution corresponds to a zero value of the denominator

$$c_P+\left(1-\frac{c_P}{R}\right)\cdot\left(\frac{R\dot{m}}{A_c}\right)^2\frac{T}{P^2} = 0.$$

This implies that

$$\dot{m} = \sqrt{\frac{c_P}{(c_P-R)RT}}PA_c, \tag{30}$$

which is the critical flow rate formula. If we eliminate flow rate in Eq. (30) with Eq. (6), then the sound speed formula follows from Eq. (30)

$$\left(\frac{P}{RT}\upsilon A_c\right)^2 = \frac{c_P}{c_\upsilon RT}P^2A_c^2,$$

or finally

$$a = \sqrt{\frac{c_P}{c_\upsilon}RT}. \tag{31}$$

Now we find out which process corresponds to the choked flow. For this purpose we substitute Eq. (30) into Eq. (15)

$$\frac{dT}{dP} = -\frac{-\frac{R}{c_p}\frac{T}{P}w_x-\frac{\lambda}{2D_h}R\frac{c_p}{(c_p-R)}\frac{T^2}{P}+f_x\frac{T}{P}}{w_x-f_x\frac{c_p}{R}+c_p\frac{c_p}{(c_p-R)}\frac{\lambda}{2D_h}T}.$$

Separating variables in the above ODE, we get

$$\frac{\frac{c_p}{gR}\left[w_x-f_x\frac{c_p}{R}+c_p\frac{c_p}{(c_p-R)}\frac{\lambda}{2D_h}T\right]}{\left[w_x-f_x\frac{c_p}{R}+c_p\frac{c_p}{(c_p-R)}\frac{\lambda}{2D_h}T\right]T}dT = \frac{dP}{P},$$

which is equivalent to

$$\frac{dT}{dP} = \frac{RT}{c_pP}.$$

Integrating the equation, we obtain

$$\frac{T}{P^{\frac{R}{c_p}}} = Const,$$

which is the Poisson adiabat. This demonstrates that the Poisson adiabat is the specific solution of Eqs. (1)–(5) at sonic speed. It also implies that in this case the flow process cannot be continuous immediately beyond the duct section where such a process occurs.

2.2.2. Solution check

We check the solution Eq. (22) of Eq. (15) written in the form

$$\frac{dP}{dz}+\left(\frac{cz}{a+cz^2}\right)P = -\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\cdot\left(\frac{a+bz^2}{a+cz^2}\right).$$

Now we substitute Eq. (23) into the above equation to obtain the ODE

$$-\frac{cz\sqrt{c}}{\sqrt{(a+c\cdot z^2)^3}}\left\{-\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{1}{\sqrt{c}}\times\left\{\frac{a}{\sqrt{c}}\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right|\right.\right.$$
$$\left.\left.+\frac{b}{\sqrt{c}}\left(\frac{z\sqrt{a+c\cdot z^2}}{2\sqrt{c}}-\frac{a}{2c}\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right|\right)\right\}+Const\right\}$$
$$-\frac{\sqrt{c}}{\sqrt{a+c\cdot z^2}}\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{1}{\sqrt{c}}\times\left\{\frac{a}{\sqrt{c}}\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right|\right.$$
$$\left.+\frac{b}{\sqrt{c}}\left(\frac{z\sqrt{a+c\cdot z^2}}{2\sqrt{c}}-\frac{a}{2c}\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right|\right)\right\}'$$
$$+\frac{\sqrt{c}}{\sqrt{a+c\cdot z^2}}\frac{cz}{a+c\cdot z^2}\left\{-\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\frac{1}{\sqrt{c}}\times\left\{\frac{a}{\sqrt{c}}\ln\right.\right.$$
$$\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right|+\frac{b}{\sqrt{c}}\times\left(\frac{z\sqrt{a+c\cdot z^2}}{2\sqrt{c}}-\frac{a}{2c}\ln\right.$$
$$\left.\left.\left.\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right|\right)\right\}+Const\right\} = -\frac{1}{R}\left(\frac{\dot{m}R}{A_c}\right)^2\left(\frac{a+b\cdot z^2}{a+c\cdot z^2}\right),$$

which is reduced after some manipulation to

$$\frac{\sqrt{c}}{\sqrt{a+c\cdot z^2}}\frac{1}{\sqrt{c}}\left\{\frac{a}{\sqrt{c}}\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right|+\frac{b}{\sqrt{c}}\left(\frac{z\sqrt{a+c\cdot z^2}}{2\sqrt{c}}-\frac{a}{2c}\ln\left|z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}\right|\right)\right\}'=\left(\frac{a+b\cdot z^2}{a+c\cdot z^2}\right).$$

Differentiation of the last equation gives

$$\frac{\sqrt{c}}{\sqrt{a+c\cdot z^2}}\frac{1}{\sqrt{c}}\left\{\frac{a}{\sqrt{c}}\frac{1+\frac{z}{\sqrt{a+c\cdot z^2}}\sqrt{c}}{z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}}+\frac{b}{\sqrt{c}}\left(\frac{\sqrt{a+c\cdot z^2}}{2\sqrt{c}}+\frac{z^2}{2\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}}-\frac{a}{2c}\frac{1+\frac{z}{\sqrt{a+c\cdot z^2}}\sqrt{c}}{z+\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}}\right)\right\}=\left(\frac{a+b\cdot z^2}{a+c\cdot z^2}\right)$$

With algebra the above equation is reduced to

$$\frac{a}{a+c\cdot z^2}+\frac{\sqrt{c}}{\sqrt{a+c\cdot z^2}}\frac{b}{c}\frac{\sqrt{a+c\cdot z^2}}{2\sqrt{c}}+\frac{\sqrt{c}}{\sqrt{a+c\cdot z^2}}\times\frac{b}{c}\frac{z^2}{2\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}}-\frac{a}{2c}\times\frac{\sqrt{c}}{\sqrt{a+c\cdot z^2}}\frac{b}{c}\frac{1}{\frac{\sqrt{a+c\cdot z^2}}{\sqrt{c}}}=\left(\frac{a+b\cdot z^2}{a+c\cdot z^2}\right)$$

from which the identity follows

$$\frac{a+bz^2}{(a+c\cdot z^2)}=\left(\frac{a+b\cdot z^2}{a+c\cdot z^2}\right)$$

that proves Eq. (22).

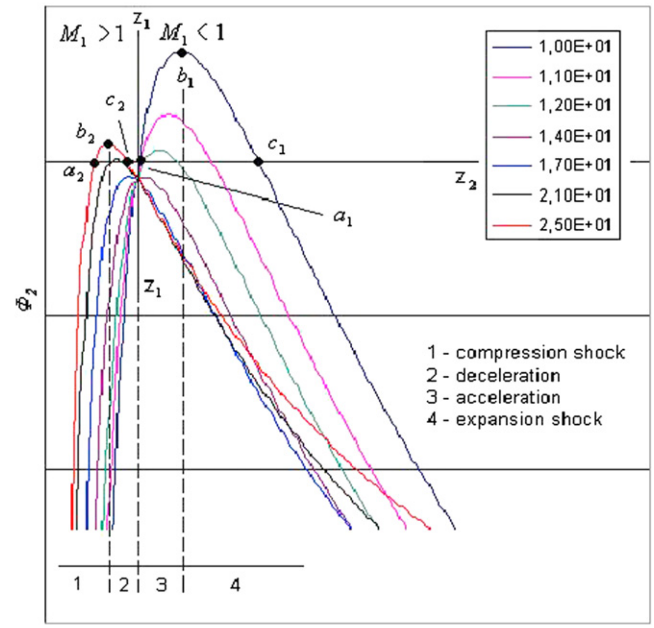

Fig. 1. Function Eq. (29).

3. Results and discussion

An analysis of the solution has two aspects which are mutually related. The first aspect concerns the identification of a type of flow; the second one concerns the computation of flow quantity values. Indeed, in order to solve Eqs. (22), (24) and (29) for duct outlet quantities it is necessary to know which type of flow occurs at its inlet. Only in this case we can obtain a unique solution, since there are generally four solutions, as illustrated in Fig. 1.

It shows the form of function $\Phi_2(z_2)$ (see Eq. (29)) for various flow rates and heat additions at a given inlet temperature and pressure.

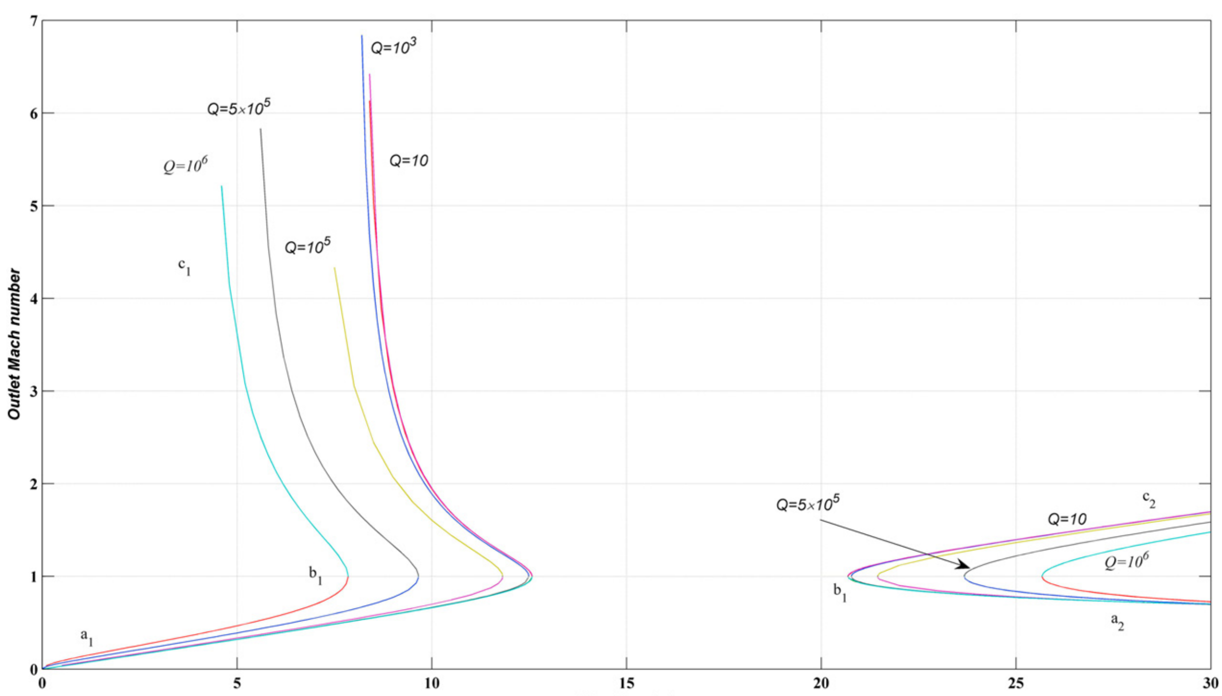

Fig. 2. Duct performance characteristic. Variation of Mach number with flow rate and heat exchange.

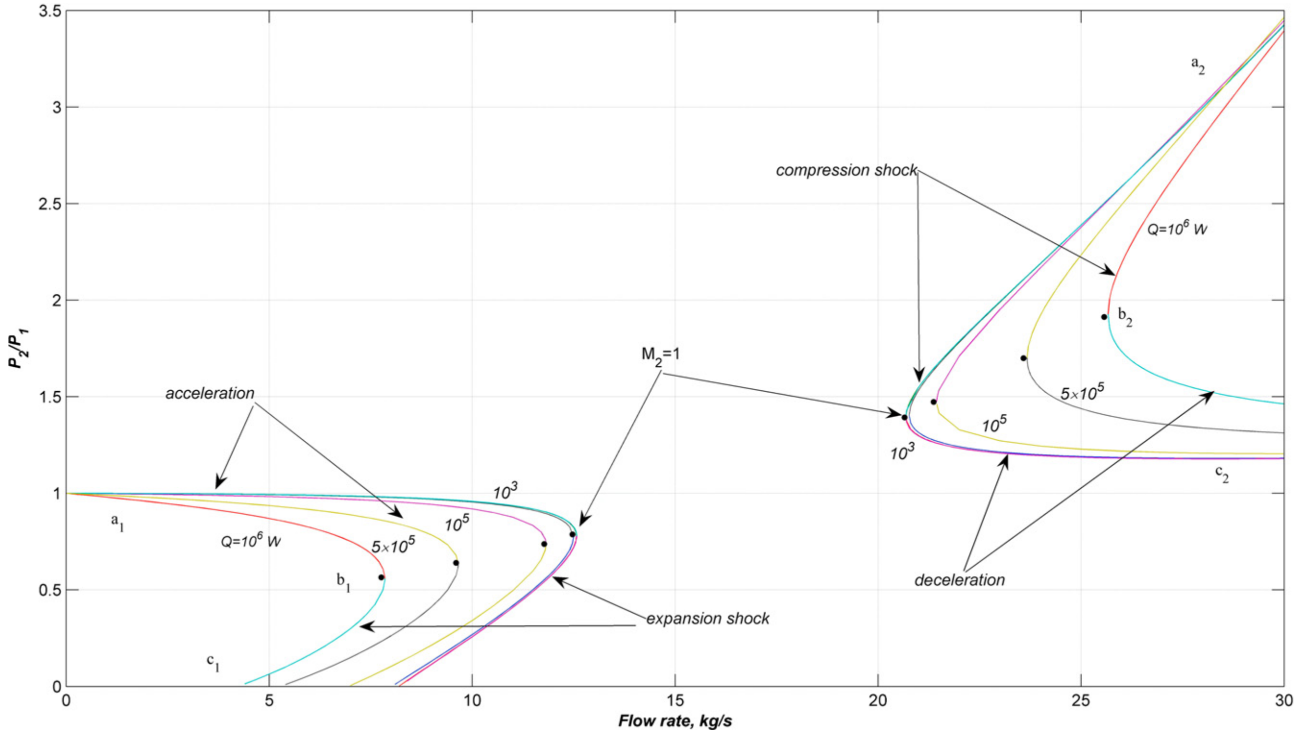

Fig. 3. Duct performance characteristic. Variation of pressure with flow rate and heat exchange.

Indeed, equation Φ_2 has either two solutions or none of the solutions for every flow rate value. Axis z_1 divides the plot into two parts. Subscripts 1 and 2 designate inlets and outlets for the flow parameters or specific points of the function $\Phi_2(z_2)$. The left and right parts correspond to flows at $M_1 > 1$ and $M_1 < 1$, respectively. These inequalities specify subsets of flow duct regimes, and may serve as the conditional transition points in a solution algorithm. To match each root to the physical process, a more careful analysis is required. For this purpose, duct performance characteristics are computed (Figs. 2–5). In Figs. 1–5, points b_1 and b_2 specify outlet choked flows ($M_2 = 1$) at $M_1 < 1$ and $M_1 > 1$, respectively. The sets of roots described by the curves a_1b_1 and a_2b_2 belong to domains $M_1 < 1$ and $M_1 > 1$, respectively. We write a for the root type when derivative $\Phi_2' > 0$, and b when $\Phi_2' < 0$. Thus, the root has subscript 1 if $M_1 < 1$ and 2

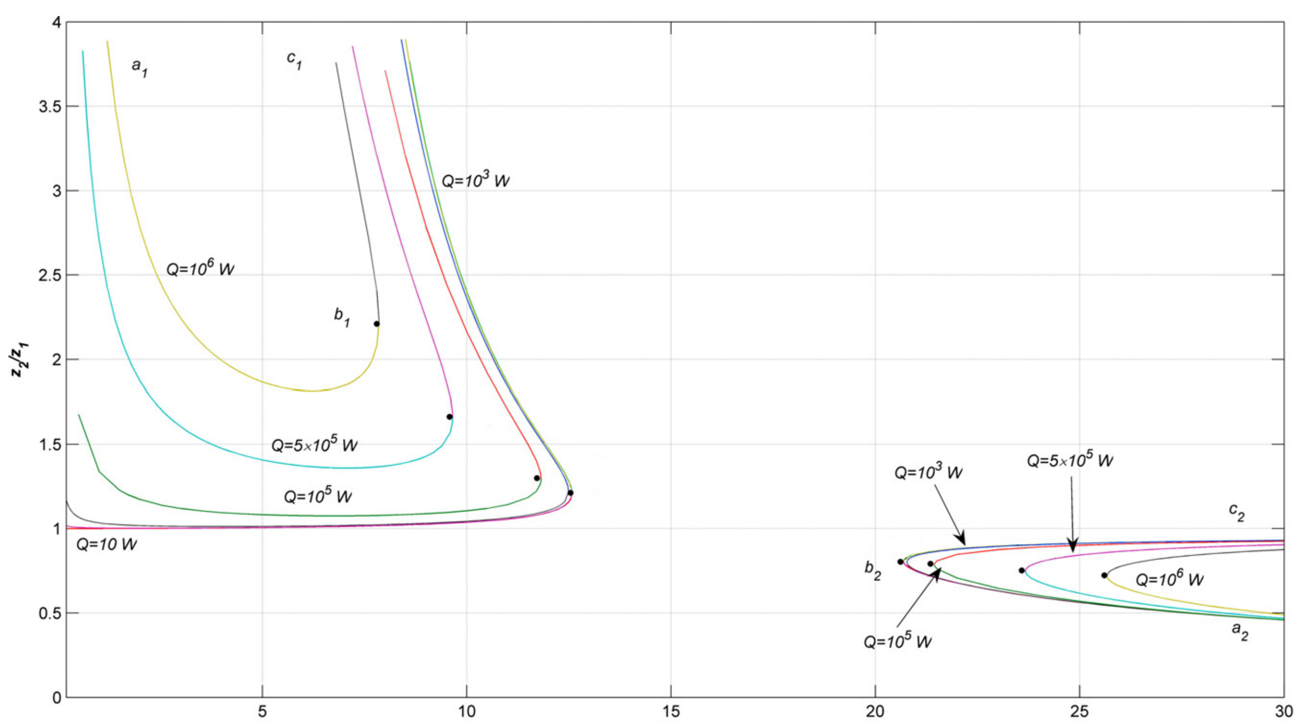

Fig. 4. Duct performance characteristic. Variation of z_2 with flow rate and heat exchange.

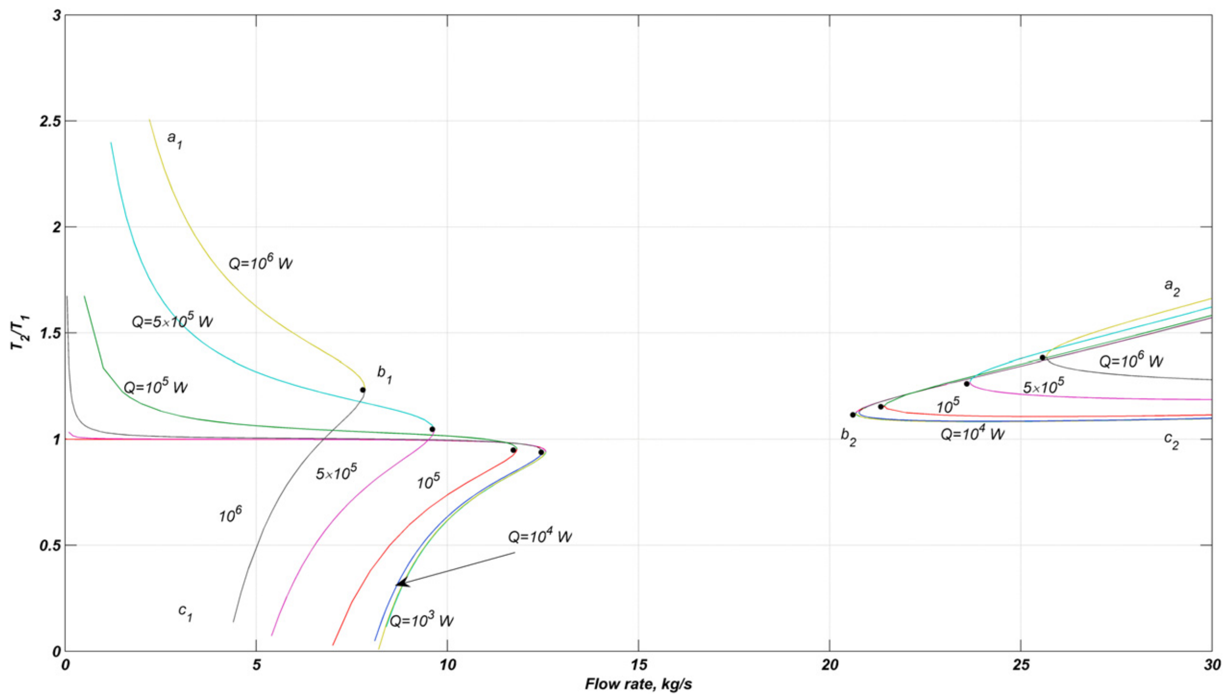

Fig. 5. Duct performance characteristic. Variation of temperature with flow rate and heat exchange.

otherwise. The curve a_1b_1 describes a subsonic flow which results in a temperature and pressure decrease at the duct outlet when flow rate increases. The heat transfer is greater, the outlet pressure and critical flow rate are less. Along this curve the Mach number increases from the duct inlet to the duct outlet, i.e. the flow accelerates (Fig. 2). Curve a_2b_2 is a result of solving the roots of Φ_2 which are at the left side of the vertical z_1 (Fig. 1). This curve describes a set of flows when $M_1 > 1$, with compression shocks at the duct outlet. The limiting point of the curve is b_2 where $M_2 = 1$. Fig. 2 shows that after the compression shock $M_2 < 1$, pressure and temperature increase (Fig. 3, 5), whereas the specific volume decreases (Fig. 4). Therefore, the temperature after the compression shock increases more slowly than pressure does. From the duct performance characteristics it becomes clear that curve b_2c_2 also describes solutions when $M_1 > 1$ provided that $M_2 > 1$, such that $M_2 < M_1$. These conditions specify a supersonic flow, which decelerates. In this case, the outlet pressure and temperature are higher than those at the inlet, but lower than in the case of a compression shock, although a specific volume is higher in the case of the supersonic deceleration. When increasing heat transfer, the choked flow rate grows (Fig. 2). Pressures and temperatures grow, but specific volumes drop at the duct outlet (see loci of bold points in Figs. 3–5). The curve b_1c_1 (Fig. 2) demonstrates the transition of the inlet subsonic flow to the outlet supersonic. This curve is obtained by solving roots of type c_1 (Fig. 1) for a set of flow

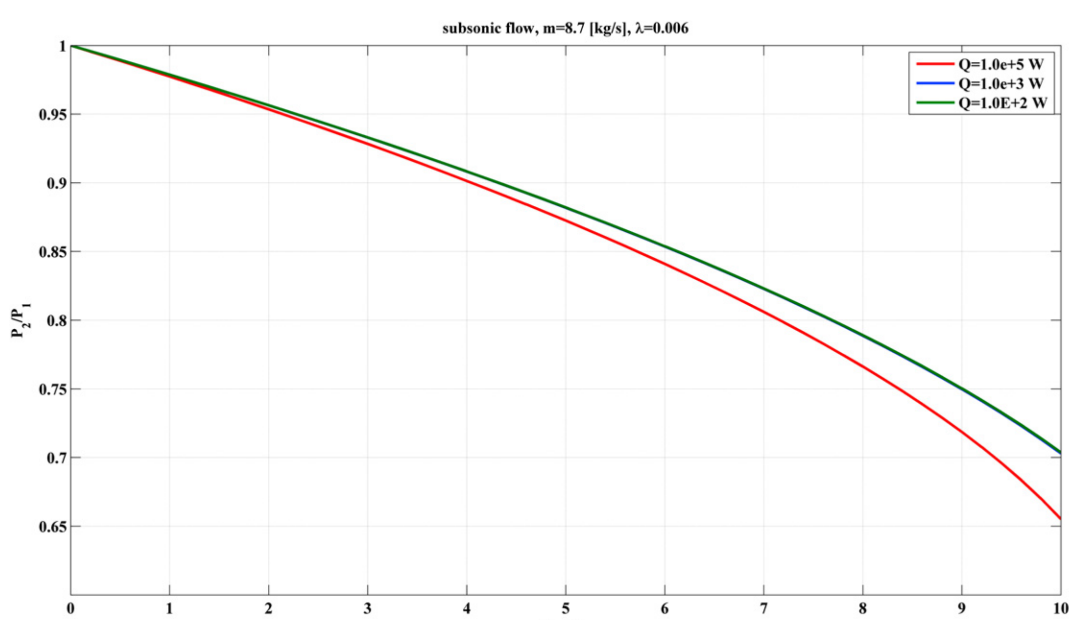

Fig. 6. Pressure distribution of subsonic flow.

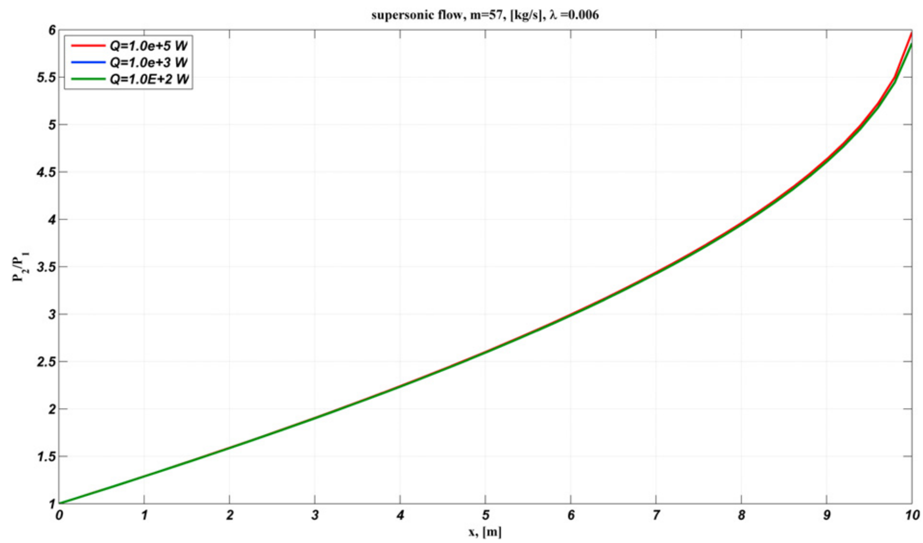

Fig. 7. Pressure distribution of supersonic flow.

rates, such that $M_1 < 1$. Thus, expansion shock solutions at the duct outlet follow from the analytical solution (Fig. 3).

Now we examine how to identify and isolate roots corresponding to various types of flows. When $M_1 < 1$ two roots exist, whose solutions correspond to the subsonic flow and to flow with an expansion shock (see Fig. 1). If $\Phi_2(z_{2,b1}) > 0$, then expansion shock is beyond the channel. So we compute z_{2,b_1} (i.e. we solve Eqs. (22), (29) for z_2 such that $M_2 = 1$), and given value of z_1 the interval $[z_1, z_{2,b_1}]$ is defined, where a unique solution of root a_1 exists. A further increase in the flow rate may result in an outlet flow with $M_2 = 1$, with b_1 tangent to z_2 axes in this case. At this point subsonic and expansion shock flows have a critical flow as a limiting value (Figs. 2–5). It means that we cannot exactly compute parameters at this point, but we can tune the computational algorithm to iterate along the curve a_1b_1 (see Fig. 1) as we are interested in continuous flow solutions in most cases. A further increase in the flow rate results in penetration of the expansion shock into the channel followed, by its moving to the channel inlet.

The process is defined by the following conditions $z_{2,b_1} > z_1$, Φ_2 $(z_{2,b1}) < 0$ (see Fig. 1). A further increase in flow rate makes compression shock move from the inlet to outlet channel section, the shock force decreases and as a result at point b_2 tangent to z_2 axes we have a symmetrical case at $M_2 = 1$, $M_1 > 1$ (Fig. 1). The similarity between the mathematical properties of the analytical solution of Eqs. (1)–(5) and the non-isentropic adiabatic solution can be seen from the above analysis. It follows that: 1) a pure expansion shock is thermodynamically impossible, therefore only the curve a_1b_1 adequately represents the flow process in the right half-plane ($z_2 > z_1$), 2) the best prediction of one-dimensional theory is for continuous flows (acceleration and deceleration), and 3) the shock compression computation gives approximate but acceptable results. Hence, the algorithms used to simulate simple types of flows can be easily adapted to the solution derived in this paper, including the calculation of critical length and compression shock position [2–5,13].

In Figs. 6–9 the distribution of pressures and temperatures are demonstrated for subsonic and supersonic flows. The results demonstrate that the solution correctly reflects the processes, the growth of pressure and temperature in the supersonic case, and the drop in pressure in the subsonic case. In the latter case we can see that during heat addition the temperature at first may increase,

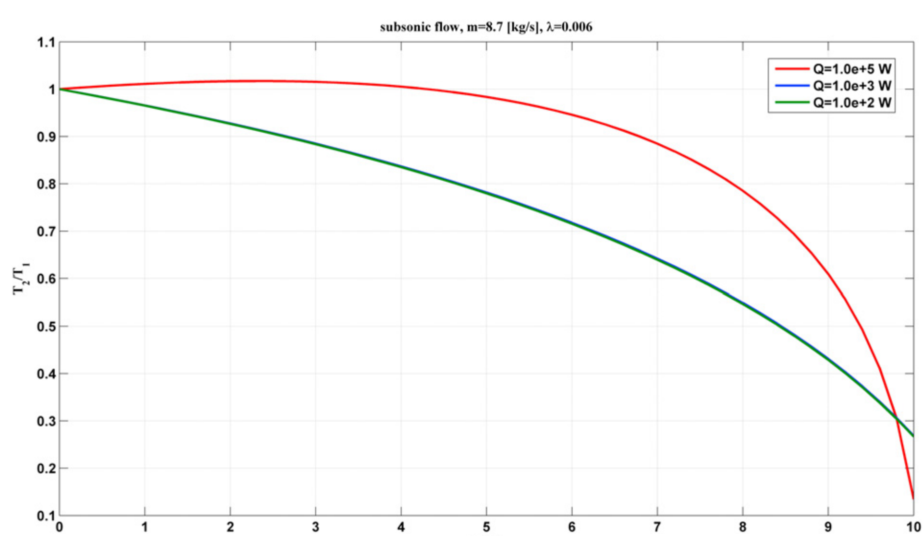

Fig. 8. Temperature distribution of subsonic flow.

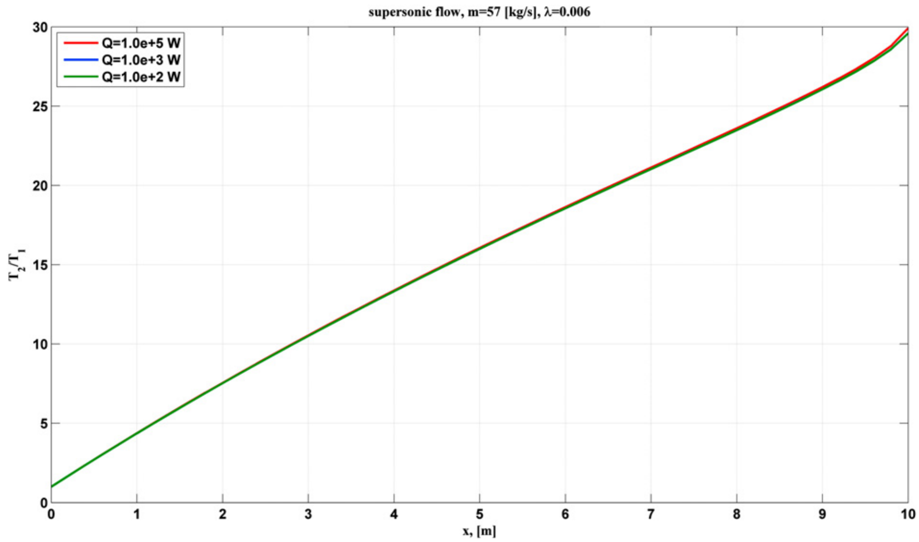

Fig. 9. Temperature distribution of supersonic flow.

but at some length it begins to decrease. This effect can be also observed in Fig. 5 as well. Moreover, heat addition to the supersonic flow is found to be ineffective.

It should be noted that heat extraction is a more complex process in the sense that the function Φ_2 is more complex to analyse as heat extraction cannot be realized in all cases in contrast to heat addition. Heat extraction is constrained by the positive values of z, otherwise the mathematical solution will give negative pressures or absolute temperatures. The heat extraction process requires an individual study, although some prerequisites follow from the solution. Allowable heat extraction is constrained by the positive values of the following root expressions of the analytical solution

$$a + c \cdot z^2 = \frac{1}{\dot{m}L}\left[q_l L + \frac{c_P}{R}\left(\frac{R}{A_c}\right)^2 \dot{m}^3 \frac{\lambda L}{2D_h} z^2\right] = \frac{1}{\dot{m}L}(\dot{Q} + \dot{W}), \quad (32)$$

where $\dot{Q}$ is heat addition, $\dot{W}$ is an equivalent heat (or a power generation), caused by friction. Eq. (32) determines the maximum amount of heat that can be extracted

$$\dot{Q} + \dot{W} > 0 \cdot \quad (33)$$

The inequality Eq. (33) must be fulfilled with respect to both inlet and outlet parameters, so it functionally depends on the solution, i.e. on z. For a duct with results shown in Figs. 6–7, i.e. at $R = 287$, $c_P = 1047$, $D_h = 0.1$, $L = 10$, $T = 573.15$, $P = 0.4905$ [MPa], $\lambda = 0.006$ and at $\dot{Q} = -104$, $\dot{m} = 6$, $M_1 = 0.5383$, the inequality Eq. (33) is satisfied for both duct sections at the following values of $T_2 = 552.43$, $P_2 = 0.3726$ [MPa], $M_2 = 0.6956$ which are the result of solution of Eqs. (22), (29).

4. Conclusions

In this paper, a general analytical solution for the model of one-dimensional steady flow of a compressible perfect gas through a constant area duct for a given constant heat flux, friction and mass force (Section 2.1) is obtained. In the case of a mass force being neglected, the particular solution of ODE (1)–(5) in terms of elementary functions is obtained by substitution method (Section 2). As a result one-dimensional analytical gas dynamics is extended with a solution in which both friction and heat flux are parameters of the flow model, whereas the analytical solutions for the simple types of flows take into account a single effect. It is demonstrated that the solution of the model in question possesses a parameter which is constant along the duct (see Eq. (22)). The parameter is not a stagnation temperature as in the case of simple types of flows. The relationship of this parameter with stagnation temperature is presented (see Eq. (27)). The stagnation temperature is not constant for the model in question, but it changes linearly along the duct (see Eq. (28)). The solution correctness is mathematically proven (Section 2.2.2). Moreover normal ODE (11)–(12), which is taken as the result of transformations of (1)–(5), has singularity that does not depend on the flow type. The singularity characterizes the choked conditions under which the Poisson adiabat formula results from Eqs. (1)–(5) as a particular solution (Section 2.2.1). The mathematical properties of the exact analytical solution were studied (Section 3). It is shown that at a given flow rate two solutions follow from Eqs. (22), (29), corresponding to a continuous flow as well as to a flow with a shock at the duct outlet (see Fig. 1). The mathematical property knowledge permitted us to compute duct performance characteristics (Figs. 2–5) as well as the distributions of temperature and pressure along the duct for both subsonic and supersonic flows (Figs. 6–9). We can see that the solution correctly models the flow processes as each result can be explained from the viewpoint of the simple types of flows. Besides, it is shown that a model with a given heat flux has the thermodynamic constraint (see Eq. (32)) that will also occur for more complex models, including numerical ones, at the same boundary conditions. If the variability of friction factor along the duct must be taken into account, then the segmentation methodology that has been already discussed (Section 1) can be suggested. In the limit of infinitely closely spaced segments a friction factor can be considered to be constant. Thus, the solution may be treated as a definition of a local heat exchange and as well as a friction factor in a duct with compressible flow from the point of view of the one-dimensional theory. So attempts could be undertaken in order to use the solution as a basis of semi-empirical models. It is very important to notice that these models will keep all the mathematical and physical properties of the solution Eq. (22), (29). The next problem associated with the application of the solution Eq. (22), (29) concerns the simulation of flow processes in the gas transportation systems. Literature reveals [16] that the ODE system Eq. (1)–(5) can also serve as a model for solving the problem with heat flux expressed by Newton's law of cooling, provided that its parameters do not change rapidly. Should we apply numerical methods or the segmentation methodology with the use of analytical models? It depends on the specific properties of the problem under solution. In any case the decision should be taken on the basis of preliminary comparative analysis (including e.g. numerical experiments) and should be guided by such a set of metrics as: problem type (e.g. analysis or design), flow system complexity, simulation time, veracity and accuracy, convergence reliability, etc. The above-mentioned problems, perhaps questionable, may be subject matter for further works on the application of the exact analytical solution obtained in the paper.

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